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Abstract

This paper extends the quadrature method of Andricopoulos, Widdicks, Duck and Newton (2003) to price exotic options under jump-diffusion models in an efficient and accurate manner. We compute the transition density of jump-extended models using convolution integrals and calculate the Greeks of options using Chebyshev polynomials. A simpler and more efficient lattice grid is presented to implement the recursion more directly in matrix form and to save running time. We apply the approach to different jump-extended models to demonstrate its universality and provide a detailed comparison of the discrete path-dependent options to demonstrate its advantages in terms of speed and accuracy.

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1 Introduction

The pricing and hedging of exotic options are of great significance to modern derivative markets and numerous methods have been derived for this purpose, including Figlewski and Gao (1999), Boyle and Tian (1999), Fusia and Meucci (2008) and Cai et al. (2015) . Extensive empirical studies have revealed that the returns of various assets have fatter tails and higher peaks than the normal distribution. For example, Bates (1996) finds that jumps in the exchange rate process are able to explain the volatility smile in the Deutsche Mark option. Johannes (2004) provides evidence of jumps in short-term interest rate models and shows that they play an important role in option pricing.

Most numerical methods, especially the tree methods, are based on diffusion models and it seems somewhat difficult to apply the tree ideas directly to the jump-diffusion process in a flexible and appealing manner, mainly because the transformation by Nelson and Ramaswamy (1990) is complex when jumps are included. Amin (1993) builds a tractable discrete time model by constructing multivariate jumps superimposed on the binomial model and Hilliard and Schwartz (2005) develop a robust bivariate tree approach. However, it is difficult to apply these ideas beyond the Black-Scholes framework or to different jump variables. Beliaeva and Nawalkha (2012) propose the mixed jump-diffusion tree; however, as pointed out in Wu et al. (2015), this is not robust when the jump size is insufficiently large. Kou and Wang (2004) and Cai and Kou (2012) derive analytical solutions for exotic options under a mixed (double)-exponential jump-diffusion model in the Black-Scholes framework. However, for most situations, analytical approximations cannot be obtained easily, especially for models outside the Black-Scholes-Merton framework.

Andricopoulos et al. (2003; 2007) and Chen et al. (2014) propose the quadrature method and deliver a novel and universal application for path-dependent options in Black-Scholes, multi-asset and volatility model cases, while also delivering exceptional accuracy and speed. In this paper, we extend the quadrature method to price exotic options under jump-diffusion models. The main gaps between diffusion and jump-extended models in the quadrature method are the estimation of the required transition density and the time-step size. To acquire a closer approximation, Chen et al. (2014) use Ait-Sahalia's algorithm to allow for a longer step up to 0.05 s or more. However, in a jump situation, this algorithm does not work and the transition density of the general jump-diffusion process calls for a multiple integral, which takes more time under a longer time step. Hence, we aim to approximate the general jump model with Merton's jump model (Merton, 1976) and a small time step is essential to achieve a closer approximation. Andricopoulos et al. (2003; 2007) focus on the convergence on the asset-price step and we pay close attention to the convergence on the time-step size.

Furthermore, as the prices of discretely monitored Asian and lookback options vary for different monitoring frequencies, a smaller time-step size will provide insight on the convergence from discretely monitored to continuously monitored frequencies. Here, we use the popular Gaussian approximation scheme to estimate the transition density of the diffusion part, which differs little from Ait-Sahalia's results when the time step is less than 0.01 s and the convolution integral is used to implement the transition density under Merton's jump model. Moreover, we present a simpler and more efficient lattice grid in which the points stay the same at each time step, thus eliminating the need for repeated calculations of transition density in recursion, leading to a substantial reduction in running time. Under the static scheme, the recursion equation is a simple matrix multiplication, the prices of exotic options are calculated directly and the hedging ratio (first and second order) can be easily estimated from the Chebyshev approximation.

The diffusion models, from both the binomial and the trinomial tree, always achieve a desirable convergence in pricing exotic options. The quadrature method outperforms the tree methods because it can correct the "distribution error" (Figlewski and Gao, 1999; Andricopoulos et al., 2003) and can handle the jump situation well, while the tree methods have limited appeal and sometimes assign the probability to nodes in an unsatisfactory way. Tree methods also need many more steps than quadrature and become more complicated when pricing Asian options and lookback options.

The remainder of this article is organized as follows. Section 2 introduces the basics of the quadrature method for jump-diffusion models, covering topics on the transition density approximation, the static lattice points we proposed and Greek calculation via the Chebyshev approximation. Section 3 discusses a universal and improved application of the technique on lookback options, multiple barrier options and Asian options under the static lattice points. In Section 4, we show the numerical results of different jump-diffusion models and put the method into perspective. Section 5 concludes the study. All of the Matlab codes used within the study are available upon request.

2 Quadrature Method

2.1 General quadrature method

In the interests of brevity, we provide only a short introduction to quadrature. A more detailed description can be found in Andricopoulos et al. (2003; 2007). Consider an option written at time 0 with expiration date T, spot price X_0 and strike price K. The model can be approximated discretely with M as equally spaced dates using the-time step $\Delta = T/M$. For path-dependent option pricing it is necessary to introduce a state variable in pricing. We take the European call lookback option as an example. At time $n\Delta\,(1 \le n \le M)$, the option value relies not only on the current price $X_{n\Delta}$ but also on the maximum price it reaches during time 0 and $n\Delta$, so we treat the maximum price as a state variable for this lookback option. The option price varies with different states.

If $V(X_{n\Delta}, \Delta, S_k)$ is the option price on node $X_{n\Delta}$ under state S_k at time $n\Delta$, the recursive equation is as follows:

$$V(X_{n\Delta}, \Delta, S_k) = \exp(-r\Delta)\int_0^{\infty} f(x \mid X_{n\Delta}, \Delta) * V(x, n\Delta + \Delta, h(S_k, x, X_{n\Delta}))dx \tag{1}$$

$n = 0,1,2,...,(M-1)$ and $k = 1,2,...,L$,

where $f(x \mid X_{n\Delta}, \Delta)$ is the conditional density of $x \mid X_{n\Delta}$ and $h(S_k, x, X_{n\Delta})$ indicates that the option price of asset price $X_{n\Delta}$ with the state S_k at $n\Delta$ relies on the value of node x with the state variable $h(S_k, x, X_{n\Delta})$ at $(n+1)\Delta$. An analytical solution to (1) is usually not available, but we can evaluate it by quadrature using the truncation of the domain where x goes from $\underline{X}_{n\Delta}$ to $\overline{X}_{n\Delta}$ instead of from 0 to positive infinity. $\overline{X}_{n\Delta}$ is the maximum price at which the underlying asset can arrive at a later time step and $\underline{X}_{n\Delta}$ is the minimum price. Sullivan (2000) proposes the Gauss quadrature because of its perfect convergence speed. Andricopoulos et al. (2003) suggest Simpson's rule to implement the task for its robustness, great convergence speed and accuracy. Here, we use Simpson's rule to complete the integral in (1) because its regularly spaced grid is convenient for pricing exotic options and estimating the recursion process. For the option with early exercisable styles, such as Bermuda, American and executive stock options, the equation for call options is as follows:

$$V(X_{n\Delta},\Delta,S_k)=\max(\exp(-r\Delta)\int_0^{\infty} f(x\,|\,X_{n\Delta},\Delta)*V(x,n\Delta+\Delta,h(S_k,x,X_{n\Delta}))dx,\max(0,S_{n\Delta}-K))$$

2.2 Transition densities

It is obvious that the transition density plays a key role in the quadrature method. The gap in the diffusion and jump-diffusion models is the transition density computation. Chen et al. (2014) use Ait-Sahalia's (1999; 2002) explicit sequence of closed-form functions to handle the diffusion model outside the Black-Scholes-Merton framework. However, in the jump situation, Ait-Sahalia's methods cannot be applied directly. Here we start from the general jump-diffusion model,

$$\mathrm{d}X_t=\mu(X_t;\theta)dt+\sigma(X_t;\theta)dW_t+d(\sum_{i=1}^{N_t} f_{N_t}(X_t,J)) \tag{2}$$

where N_t is a Poisson process with intensity parameter λ, and $f_{N_t}(X_t,J)$ is the jump function depending on X_t and the jump variable J. For a tiny time step, we can reduce the model to the following widely applicable form:

$$\mathrm{d}X_t=\mu(X_t;\theta)dt+\sigma(X_t;\theta)dW_t+f(X_t,J)dN(\lambda) \tag{3}$$

where $dN(\lambda)$ is a Bernoulli distribution, which takes the value of 1 with a probability of λdt and the value of 0 with a probability of $1-\lambda dt$. Merton (1976) introduced this Poisson-driven process for the jump process.

The PDF in (3) is as follows:

$$\mathrm{d}X_t=\begin{cases}\mu(X_t;\theta)dt+\sigma(X_t;\theta)dW_t, & \text{with probability } 1-\lambda dt\\ \mu(X_t;\theta)dt+\sigma(X_t;\theta)dW+f(X_t,J), & \text{with probability } \lambda dt\end{cases} \tag{4}$$

We are able to calculate the transition density of the first part, $\mu(X_t;\theta)dt+\sigma(X_t;\theta)dW_t$, which we easily denote as P_0 according to the Gaussian approximation scheme (a normal distribution). The second part, $\mu(X_t;\theta)dt+\sigma(X_t;\theta)dW+f(X_t,J)$, we denote as P_1 according to the convolution integral because it is a sum of two random variables:

$$P_1(x,\Delta\,|\,X_t;\theta)=\int\phi(x-y\,|\,X_t;\theta)\,p(y\,|\,X_t;\theta)dy \tag{5}$$

Usually, P_1 cannot be expressed in a closed form. Fortunately, there are a variety of numerical integral methods useful for completing the integral. As such, according to the Bayes rule, the transition density is as follows:

$$P(X_{t+1},\Delta \mid X_t;\theta) = (1-\lambda\Delta)P_0(X_{t+1},\Delta \mid X_t;\theta) + \lambda\Delta P_1(X_{t+1},\Delta \mid X_t;\theta) \quad (6)$$

In the work of Chen et al. (2014), the time-step size Δ can be up to 0.05s or more. However, in the jump-extended situation, a smaller Δ is preferred to guarantee the accuracy of density, not only for the Merton jump model close to the general jump model, but also for the Gaussian approximation scheme of P_1 close to the model. Wu et al. (2015) show that the jump-extended situation possesses good convergence in American option pricing when Δ is less than 0.01 s.

2.3 Lattice points

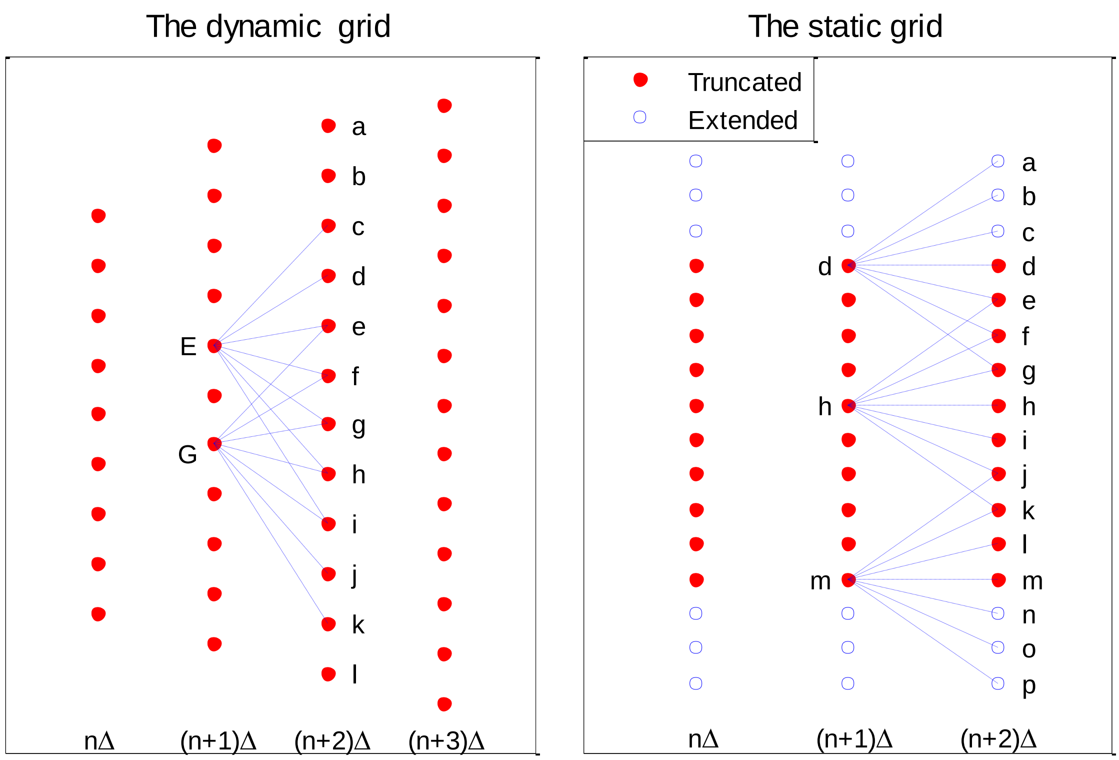
When a smaller time step size is selected, more recursion steps and more calculations of transition densities are required. Thus, it is necessary to construct the lattice points more effectively than in Andricopoulos et al. (2003; 2007). In doing so, let $\bar{A}_{n\Delta}(n=0,1,\ldots,M)$ be the maximum price that the asset can reach at each observation point, and let $\underline{A}_{n\Delta}(n=0,1,\ldots,M)$ be the minimum price at each observation point. In contrast to the dynamic (time-varying) lattice points in Andricopoulos et al. (2003; 2007), we choose the static nodes suggested in Wu et al. (2015), which means that $X^i_{n\Delta} = X^i_{M\Delta}, \underline{A}_{n\Delta} = \underline{A}, \bar{A}_{n\Delta} = \bar{A}\ (0 \le n \le M)$. Let δ be the price-step size in the grids. The vector of lattice points we construct before expiration is as follows:

$$((-I_{\min}):1:I_{\max})*\delta + X_0$$

where $I_{\min}$ is the nearest integer less than or equal to $(X_0-\underline{A})/\delta$, $I_{\max}$ is the nearest integer greater than or equal to $(\bar{A}-X_0)/\delta$ and $(-I_{\min}):1:I_{\max}$ is a regularly spaced vector. By adopting the static node scheme and initially calculating the transition density matrix, we avoid repeated time-consuming computations in recursion and we see a substantial reduction in running time. Thus, without the time loss on the transition density computation, a smaller time step and more accurate transition density are both possible. The second advantage of a static node scheme over a dynamic one is that we can calculate the option price on different spot prices immediately at each observation point, and it is easy to derive the option Greeks, especially Delta, Gamma, Theta and Charm, using the Chebyshev approximation. However, within the dynamic framework, the numbers of nodes and the scale they cover decrease as we move backward, so we need to calculate the values node by node. The third advantage is that it is much easier to price the exotic options now that we are able to accomplish the recursion in (1) using only a matrix multiplication. This avoids an exponential increase in the effort involved in calculating the values as we move further backward and we remove the extrapolation, which may take extra time, as reported in Andricopoulos et al. (2003; 2007) and Chen et al. (2014). This is because the states in a static point scheme merely present a linear increase for Asian options and no increase for multiple-barrier and lookback options, which we discuss later.

For most options within 0.5 years, we take $\bar{A}$ to be $2X_0$ (or $1.8X_0$) and $\underline{A}$ to be $X_0/2$ (or $X_0/3$). Wu et al. (2015) use a similar extension for nodes to guarantee the accuracy of points near $\underline{A}$ or $\bar{A}$. Andricopoulos et al. (2003; 2007) and Chen et al. (2014) show that the smaller the price step, the more accurate the options. From our experience, it reaches a penny accuracy when δ is less than 0.2 for the case $X_0 = 100$ and an appropriate parameter is set in the model.

Figure 1 Comparison of dynamic and static grids



2.4 Option hedging

To approximate Delta $\Delta(X_0)$, which is the sensitivity of the option price with respect to changes in the asset price, the simplest approach is to use a finite difference approximation. We can obtain the option price $V(X_0+\varepsilon)$ and $V(X_0-\varepsilon)$ with a spot price equal to $X_0+\varepsilon$ and $X_0-\varepsilon$ (let ε be a number close to zero), then $\Delta(X_0)$ can be estimated by

$\Delta(X_0) \approx (V(X_0+\varepsilon)-V(X_0-\varepsilon))/(2\varepsilon)$. Although the expense of performing a resimulation for different spot prices is obviously disadvantageous, other universal methods have some drawbacks that limit their use in practice. The simulation-based likelihood ratio method (see Glasserman, 2003) may face a larger variance, and the WD method (Pflug and Weisshaupt, 2005) needs a time-consuming simulation. Our method seems to be much easier. Based on the grid scheme in the previous subsection, we can calculate the prices on nodes *d* to *m* in Figure 1 at time 0. As mentioned in Sullivan (2002), function approximation can be used to acquire a close and satisfactory fit for these nodes. Thus, hedging coefficients can be easily computed with the fit curve estimated. Here, we select the Chebyshev polynomials as the candidate approximation and the first-order and second-order derivative can be derived directly.

3 Specific option pricing

The Bermuda option was considered in Andricopoulos et al. (2003) and the American option was solved by Wu et al. (2015). Here, we focus on the lookback option, multiple barrier option and Asian option under the simple lattice grid.

3.1 Lookback option

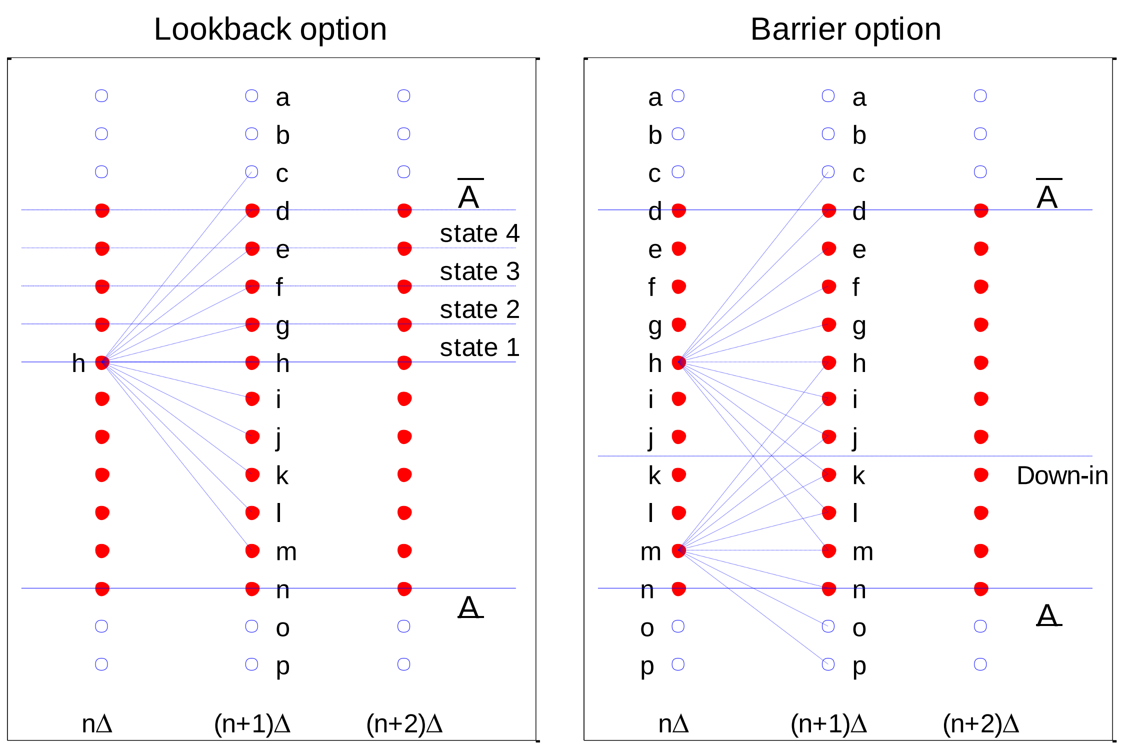
First, it is necessary to pay close attention to the lookback option, whose payoffs depend on the maximum or minimum price it reaches before expiration. Both floating and fixed strike options are available. Here, we show an example of a call lookback option priced at time $n\Delta$ on a particular node. The state variables are the maximum prices it reaches by time $n\Delta$ at node *h*: *a, b, c, d, e, f, g, h.* The value of *h* in state *f* at time $n\Delta$ is calculated from the value of nodes *f* to *n* in state *f* and the value of nodes *a* to *e* in their own state at time $(n+1)\Delta$. We keep in mind that the state variables stay the same through the option lifetime under a static grid and, as highlighted in Andricopoulos et al. (2007), help to prevent the non-linearity error. At each time step, there is an option value matrix $U_{n\Delta}$ that contains the values for all of the nodes in different states. The following simple equation represents the recursion equation (1):

$$V_{n\Delta} = \exp(-r\Delta) * \delta * \text{Had}(W, P) * U_{(n+1)\Delta} \tag{7}$$

where W represents the weight coefficient matrix in applying Simpson's rule, $P(i, j)$ is the conditional density of $X_{n\Delta}^{j} \mid X_{n\Delta}^{i}$ in one time step and *Had* refers to the Hadamard product for matrix W and P. $U_{(n+1)\Delta}(i,j)(i \le j)$ is the option value of X_i in state $S_j(S_j = X_j)$ at time $(n+1)\Delta$ and $U_{(n+1)\Delta}(i,j) = V_{(n+1)\Delta}(i,j)$ for $i \le j$ and $U_{(n+1)\Delta}(i,j) = V_{(n+1)\Delta}(i,i)$ for $i > j$.

$U_{M\Delta}$, which is an upper triangular matrix at maturity, is the result and we complete the recursion through (7) step by step. The resulting option values of nodes *d* to *n* under different states at time 0 and the numbers in the main diagonal are the option prices. The only difference between floating and fixed strike options lies in the value matrix $U_{M\Delta}$ at maturity, which is easy to find. Once the option is valued for different spot prices, applying the Chebyshev approximation and computing the derivative of the fitted polynomials is relatively easy.

Figure 2 The static grid for lookback and barrier options



3.2 Multiple barrier options

The payoff for a barrier option depends on whether the price crosses a certain level, either knocked out or knocked in. A knock-out option can be easily valued by the quadrature method because it is worth nothing when the price moves beyond the barriers. The value of a knock-in call equals the value of a regular call minus the value of a knock-out call. Therefore, the knock-in option is not difficult to calculate. However, more work seems necessary to calculate multiple barrier options. Here, we demonstrate how to value multiple barrier options directly using the quadrature method.

We highlight the case of a down-and-in call barrier option price. There are two states, which relate to whether or not it is knocked in up to time $n\Delta$ at *h*. As depicted in Figure 2, the value of *h* in the knocked-in state at time $n\Delta$ is calculated from the value of nodes *c* to *m*, which are in the knocked-in state at time $(n+1)\Delta$. The value of *h* in the state that is not yet knocked-in at time $n\Delta$ is calculated from the value of nodes *c* to *j*, which are not yet knocked-in, plus *k* to *m*, which are in the knocked-in state at time $(n+1)\Delta$. The value of *m* at time $n\Delta$ is calculated from nodes *h* to *p* in their knocked-in states. Thus, we find that it is necessary to record the value of the nodes in two states at each time step. It is not difficult to complete the aforementioned algorithm in a matrix (vector) form.

When we move forward to a double knock-in barrier option that is available once either (or both) of these barriers is breached, then there are four states (both knocked-in, one or the other knocked-in, not yet knocked-in) to record for every node and the procedure can be worked out directly with fewer states compared with the lookback option. Similarly, it is not difficult to extend the idea to the moving barriers or early exercise barrier option.

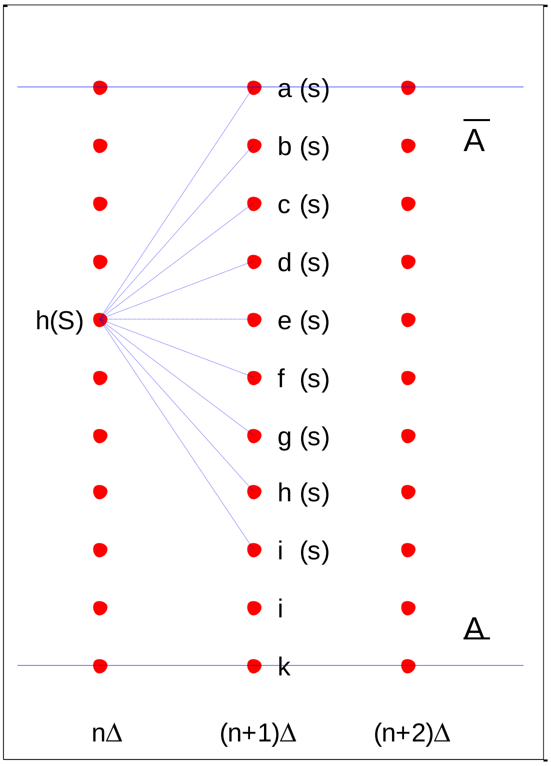
3.3 Asian options

In this section we investigate the problem of discretely monitored Asian options under the static grid scheme. Asian options provide a payoff at maturity, based on the arithmetic (geometric) average before the expiration date. Floating strike options and fixed strike options also exist. For most work on Asian option pricing, the models are built under the Black Scholes assumption as they are analytically tractable and provide a closed-form solution in terms of the Fourier transformation (see Benhamou (2002), Fusai and Meucci (2008) and Cai and Kou (2012) for examples of this). Numerical methods include the tree method of Hull and White (1993), the quadrature method with Chebyshev approximation in Andricopoulos et al. (2007) and a convolution structure of recursion under Lévy processes as proposed by Fusai and Meucci (2008).

We present a method based on transition density to price Asian options, taking the pricing of the arithmetic average Asian options with fixed strike and a payoff of $\max(\frac{1}{M}\sum_{i=0}^{M} X_{i\Delta} - K, 0)$ as an example. The state variable for this option at time $n\Delta$ is the sum of prices from maturity T

backward to $n\Delta : S_{n\Delta} = \sum_{i=n}^{M} X_{i\Delta}$. At time $n\Delta$, the state variable is an arithmetic progression from $(M-n+1)*\underline{A}$ to $(M-n+1)*\overline{A}$ with a constant difference of δ, which contains $N_{n\Delta} = (M-n+1)*(I_{\min}+I_{\max})$ elements and presents a linear increase. Separately from recursion equation (1), we recursively compute the transition probability matrix of the state variables for all nodes at time $n\Delta$.

Figure 3 The static grid for Asian options



$$p(X_{n\Delta}, \Delta, S_{n\Delta,k}) = \int_0^{\infty} f(x \mid X_{n\Delta}, \Delta) * p(x, n\Delta + \Delta, S_{n\Delta,k} - X_{n\Delta})dx \tag{8}$$

Again, we approximate the integral (9) discretely via the following matrix form:

$$Q_{n\Delta} = U_{(n+1)\Delta} * P * \delta X \tag{9}$$

where $P(j,i)$ is the conditional density of $X_{n\Delta}^{j} \mid X_{n\Delta}^{i}$ in one time step and $U_{(n+1)\Delta}(j,i)$ is the probability matrix of X_i in state $S_{(n+1)\Delta,j}(S_{(n+1)\Delta,j} = (M-n+1)*\underline{A} + (j-1)*\delta X, 1 \le j \le N_{n\Delta})$ at time $(n+1)\Delta$ and $U_{(n+1)\Delta}(j,i) = Q_{(n+1)\Delta}(j-i+1,i), j > i$. At maturity, $U_{M\Delta}$ is the identity matrix. This method is similar to that of Benhamou (2002) and Fusai and Meucci (2008), who discuss it within the Black-Scholes framework. Our method extends the idea beyond the Black-Scholes framework and it seems relatively easy to complete (10). Finally, at time 0, we get an approximation of the distribution of $S_0 = \sum_{i=0}^{M} X_{i\Delta}$, which is of course conditional on X_0. Consequently, the valuation of the option and estimation of the Greeks are easy to derive. As for floating average Asian options, we take $S_{n\Delta} = \sum_{i=n}^{M} X_{i\Delta} - (M+1)X_{M\Delta}$ as the corresponding state variable and use double the number of fixed cases. When turning to the geometric average Asian options under the static grid scheme, which have a payoff of $\max(\exp(1/(M+1)\sum_{i=0}^{M} \log(X_{i\Delta})) - K, 0)$, we are able to calculate the transition density of $P(\log(X_{(n+1)\Delta}), \Delta \,|\, \log(X_{n\Delta}); \theta)$ through $P(X_{(n+1)\Delta}, \Delta \,|\, X_{n\Delta}; \theta)$ and treat $S_{n\Delta} = \sum_{i=n}^{M} \log(X_{i\Delta})$ as the state variable. The remaining calculation stays the same with arithmetic average Asian options.

The linear increase in state variables may still require a large-scale matrix once we have hundreds of steps. Fortunately, our results in the subsequent section indicate perfect convergence on the asset-price steps. In practice, the points near $\underline{A}$ or $\bar{A}$ can be "absorbed" into truncated points. Furthermore, based on (10), the truncation of the domain for each node ranges from $\underline{A}$ to $\bar{A}$ instead of five or six standard deviations away from node $X_{n\Delta}$ in a time step . This also eliminates the need for the extrapolation procedure in Andricopoulos et al. (2007), which requires more effort.

4 Numerical results

In this section, we consider exotic options under a wide range of jump-diffusion models to illustrate the performance of the quadrature method. Specifically, we select the lognormal jump-extended CEV model, the exponential jump-extended DFW model (see Dumas, Fleming and Whaley, 1998) and the double-exponential jump model (see Kou, 2002, and Kou and Wang, 2004) as representatives. Due to the limited appeal of tree methods in jump-extended models, we treat the Monte Carlo results as benchmarks.

4.1 Lookback options under the lognormal jump-extended CEV model

To check the accuracy of the quadrature method, we test several cases under the lognormal jump-extended CEV model:

$$\mathrm{d}X_t = rX_t dt + \sigma X_t^{\rho} dW_t + X_t(\exp(J) - 1) dN(\lambda)$$

where X_t is the underlying asset price and $J \sim N(\mu_J, \sigma_J^2)$. The lognormal distribution jump is derived by Johannes (2004) into continuous-time interest rate models, allowing both negative and positive jumps and avoiding the interest rates becoming negative. By the Gaussian approximation scheme, we have the following density:

$$P_0(x_{t+\Delta}, \Delta \mid x_t; \theta) = \frac{1}{\sqrt{2\pi\Delta}\sigma x_t^{\rho}} \exp\left(-\frac{(x_{t+\Delta} - \alpha x_t \Delta - x_t)^2}{2(\sigma x_t^{\rho})^2 \Delta}\right) \tag{10}$$

$P_1(x_{t+1}, \Delta \mid x_t; \theta)$ is expressed in the following convolution integral:

$$\begin{aligned} P_1(x_{t+\Delta}, \Delta \mid x_t; \theta) = \int_{-x_t}^{+\infty} & \frac{1}{\sqrt{2\pi\Delta}\sigma x_t^{\rho}} \exp\left(-\frac{(x_{t+\Delta} - x - \alpha x_t \Delta - x_t)^2}{2(\sigma x_t^{\rho})^2 \Delta}\right) \\ & * \frac{1}{\sqrt{2\pi}\sigma(x + x_t)} \exp\left(-\frac{1}{2}\left(\frac{\ln(x/x_t + 1) - \mu_J}{\sigma_J}\right)^2\right) dx \end{aligned} \tag{11}$$

Then,

$$P(x_{t+\Delta}, \Delta \mid x_t; \theta) = (1 - \lambda\Delta) P_0(x_{t+\Delta}, \Delta \mid x_t; \theta) + \lambda\Delta P_1(x_{t+\Delta}, \Delta \mid x_t; \theta)$$

We use Simpson's rule to implement convolution integral (12). We can calculate the 0.05% and 99.95% percentiles of the jump variable $X_t(\exp(J) - 1)$ and establish an equally spaced grid with 200 or more quadratic points between the percentiles. From this the integral can be numerically approximated using Simpson's rule. We choose an appropriate parameter setting for the CEV models, $r = 0.05, \rho = 0.9, \mu_J = 0.02, \sigma_J = 0.03$, and the maturity is 0.5 years and $X_0 = 100$.

In Table 1, we collect the prices for different strike prices, jump intensities and volatilities. The average absolute error is around 0.005. The time taken to generate one numerical result by quadrature is around 5 seconds for $\Delta = 1/200\,(400)$, which significantly outperforms the Monte Carlo method. Table 2 presents the Greek letter Delta in the lookback option. Estimates from the Chebyshev approximation are also similar to the results produced by the Monte Carlo method (finite difference).

Table 1 Lookback option prices for the lognormal jump-extended model

Strike	σ	λ	Quad-200	MC-200	SE	Quad-400	MC-400	SE
	0.30	1	8.699	8.701	0.010	8.931	8.934	0.010
Floating	0.30	3	8.244	8.248	0.010	8.482	8.492	0.010
	0.40	1	12.169	12.175	0.013	12.488	12.492	0.013
100	0.30	1	12.204	12.210	0.014	12.436	12.436	0.014
	0.30	3	13.853	13.863	0.016	14.092	14.097	0.016

	0.40	1	15.671	15.683	0.019	15.990	15.992	0.019
	0.30	1	5.070	5.071	0.011	5.200	5.195	0.011
110	0.30	3	6.370	6.371	0.013	6.518	6.517	0.013
	0.40	1	8.158	8.164	0.016	8.365	8.363	0.016
Running Time			4.8s	13.8s		5.3s	29.0s	

The columns "Quad-200 (400)" stand for $\Delta = 1/200(400)$ and the same is true of MC-200(400). Quad-200 (400), MC-200 (400) and SE denote the quadrature results, the Monte Carlo simulation estimates and associated standard errors based on 50,000 (25,000*2) replications with the antithetic variates method.

Table 2 Estimating the Delta of the lookback options

Strike	σ	λ	Quad-400			MC-400		
			90	95	100	90	95	100
	0.30	1	0.0801	0.0795	0.0790	0.0798	0.0793	0.0788
Floating	0.30	3	0.0759	0.0753	0.0748	0.0756	0.0751	0.0747
	0.40	1	0.1205	0.1146	0.1117	0.1118	0.1111	0.1105
	0.30	1	0.5616	0.8197	1.0649	0.5618	0.8187	1.0738
100	0.30	3	0.6365	0.8743	1.0849	0.6367	0.8739	1.0930
	0.40	1	0.6823	0.8937	1.0944	0.6834	0.8933	1.1038

This shows the estimated value of Delta for discretely monitored lookback options with $\Delta = 1/400$, using the Chebyshev approximation and finite difference method under different strike prices for several spot prices. ε is equal to 0.0025.

4.2 Multiple Barrier options

To demonstrate its universality, we consider a double barrier call option (Down-and-out and Up-and-in) under the DFW model with an exponential jump variable that is introduced by Duffie et al. (1999) to suggest a positive spike for stock return volatility:

$$\mathrm{d}X_t = rX_t dt + \sigma X_t (X_t + b) dW_t + JdN(\lambda)$$

where $J \sim \exp(1/\eta)$ and $P(x_{t+1}, \Delta \mid x_t; \theta)$ are easy to calculate in the same way as (11) and (12). The parameters used in our numerical example are $r = 0.05, b = 0, X_0 = 100, t = 0.5$, written with a knock-out barrier of 90 and a knock-in barrier of 110. Table 3 reports the prices and the running times. We show that our method can reach penny accuracy within different strikes, volatilities and jump intensities. Furthermore, they can be obtained within one second when Δ is equal to $1/200\,(400)$. We find that all of the quadrature values are located within 95% confidence intervals of the associated MC values and one barrier option price generated from quadrature saves a lot of time compared with the Monte Carlo method.

Table 3 Double barrier option prices

Strike	σ	λ	η	Quad-200	MC-200	SE	Quad-400	MC-400	SE
90	0.0015	1	4	10.751	10.751	0.019	10.848	10.831	0.019
	0.0015	3	4	15.046	15.049	0.022	15.125	15.113	0.022
	0.0015	3	2	11.724	11.710	0.019	11.825	11.799	0.019
	0.0020	1	4	12.298	12.314	0.022	12.325	12.313	0.022
100	0.0015	1	4	6.305	6.311	0.014	6.334	6.327	0.013
	0.0015	3	4	9.445	9.451	0.017	9.467	9.460	0.017
	0.0015	3	2	6.926	6.925	0.014	6.957	6.943	0.014
	0.0020	1	4	7.618	7.632	0.017	7.614	7.603	0.017
110	0.0015	1	4	2.657	2.663	0.009	2.657	2.658	0.009
	0.0015	3	4	4.634	4.642	0.012	4.634	4.628	0.012
	0.0015	3	2	2.938	2.941	0.009	2.939	2.932	0.009
	0.0020	1	4	3.930	3.939	0.013	3.924	3.918	0.013
Running time (s)				0.3	6.7		0.4	13.9	

The table compares the quadrature method with the Monte Carlo method. The meaning of Quad-100, Quad-200, MC-200 and MC-400 are the same as in Table 1.

4.3 Asian options under HEM

In this subsection, we price Asian options under the double-exponential jump diffusion model (Kou, 2002):

$$\frac{dX(t)}{X(t-)} = \mu dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(V_i - 1))$$

where $y = \ln(V_i)$ has the PDF $f(y) = p*\eta_1*\exp(-\eta_1*y)*1_{(y\geq 0)} + (1-p)*\eta_2*\exp(\eta_2*y)*1_{(y<0)}$. An extension to this model is the mixed-exponential jump model (see also Cai and Kou, 2011), which is general enough to approximate any jump-size distribution. The explicit density over a time interval Δ can be found in Kou (2002). A parameter set is given by $\mu = 0.05, \eta_1 = \eta_2 = 30$ and $X_0 = 100$. We consider the arithmetic average of Asian call options with various strikes, jump intensities, volatilities and maturities. Table 4 provides the results of fixed strike price options and Table 5 presents the Delta estimates. As mentioned in Andricopoulos et al. (2007), when Chebyshev approximating polynomials are required, quadrature does not produce remarkable results for Asian options when there are many observations. Our results show that the convergence speed on the price step is fast, and impressive accuracy is again attained on both the price and the Delta. Table 6 shows that the running time and quadrature save a lot of time compared with the Monte Carlo method when the step is equal to 1 or Δ is equal to 1/100.

Table 4 Asian options under the double exponential jump diffusion model

Strike	σ	T	λ	Quad-100 step=1.0	Quad-100 step=0.5	MC-100	SE	Quad-200 step=1.0	Quad-200 step=0.5	MC-200	SE
90	0.20	0.50	1	11.104	11.104	11.101	0.011	11.105	11.105	11.104	0.011
	0.20	0.50	3	10.873	10.873	10.871	0.011	10.876	10.876	10.869	0.011
	0.20	1.00	1	12.364	12.364	12.368	0.015	12.365	12.365	12.369	0.015
	0.30	0.50	1	11.863	11.863	11.860	0.015	11.867	11.867	11.864	0.015
100	0.20	0.50	1	3.824	3.824	3.825	0.008	3.832	3.832	3.830	0.008
	0.20	0.50	3	3.796	3.796	3.800	0.008	3.804	3.804	3.797	0.008
	0.20	1.00	1	5.662	5.662	5.666	0.011	5.667	5.667	5.670	0.011
	0.30	0.50	1	5.378	5.378	5.382	0.012	5.389	5.389	5.387	0.012
110	0.20	0.50	1	0.714	0.714	0.716	0.003	0.721	0.721	0.721	0.003
	0.20	0.50	3	0.758	0.758	0.761	0.004	0.765	0.765	0.763	0.004
	0.20	1.00	1	1.974	1.974	1.982	0.007	1.980	1.980	1.986	0.007
	0.30	0.50	1	1.892	1.892	1.896	0.007	1.905	1.905	1.905	0.007

The columns “Quad-100 (200)” represent $\Delta = 1/100\,(200)$ and the same is true of MC-100 (200). “step = 1.0(0.5)” means the price step is 1(0.5). MC-200 (400) and SE denote the Monte Carlo simulation estimates and associated standard errors based on 50,000 (250,000 plus 250,000 antithetic) replications.

Table 5 Estimating the Delta of the Asian options

Strike	σ	T	λ	Quad-1 90	Quad-1 100	Quad-1 110	Quad-0.5 90	Quad-0.5 100	Quad-0.5 110	MC 90	MC 100	MC 110
100	0.20	0.50	1	0.1367	0.5580	0.8930	0.1368	0.5580	0.8930	0.1365	0.5579	0.8933
	0.20	0.50	3	0.1390	0.5409	0.8738	0.1390	0.5409	0.8738	0.1385	0.5411	0.8740
	0.20	1.00	1	0.2510	0.5754	0.8268	0.2510	0.5754	0.8268	0.2506	0.5752	0.8282
	0.30	0.50	1	0.2369	0.5472	0.8073	0.2370	0.5472	0.8073	0.2372	0.5472	0.8081

This table compares the Delta from the Chebyshev approximation by the quadrature method and the finite difference by the Monte Carlo method with different spot prices under a strike price of 100. Quad-1 (0.5) indicates that the price step was set at 1 (0.5). The MC results are from 500,000 paths (250,000 plus 250,000 antithetic) and the ε value is 0.0025.

Table 6 Running time (in seconds)

Δ	Quad-1	Quad-0.5	MC
1/100	0.9	3.8	4.8
1/200	3.2	14.0	9.5

This table compares the running time with different time-step sizes. The columns "Quad-1 (0.5)" stand for δ =1(0.5).

5 Conclusion

In this paper, we extend the quadrature method (Andricopoulos et al., 2003) to jump-diffusion models. Our paper makes three contributions. First, we generalize the efficient quadrature method to jump-diffusion models with the transition densities calculated from convolution integrals and we allow a smaller time step. Second, we propose a simpler and more robust static grid and complete the recursion in matrix form. Third, we estimate both first- and second-order hedging ratios via the Chebyshev approximation based on the option prices. The calculation is simple and precise. We run numerical tests on exotic options, including lookback options, multiple-barrier options and Asian options, and these examples show that our method offers good accuracy and greatly reduced running times compared with the popular Monte Carlo method.

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