北京大学中国经济研究中心
China Center for Economic Research

讨论稿系列
Working Paper Series

No.E2016010 September 2016

Pricing American-Style Options under Jump-Diffusion Models by the Quadrature Method*

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Abstract

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* Yong Li (corresponding author) gratefully acknowledges the financial support of the Fundamental Research Funds for the Central Universities and the Research Funds of Renmin University of China (No. 14XNI005). Zhuo Huang acknowledges the financial support of the National Science Foundation of China (No. 71201001, 71671004). Corresponding email: gibbsli@ruc.edu.cn. We are grateful to David P. Newton for sharing his codes.

1. Introduction

American-style option valuation is of great significance in finance. Numerous methods of pricing American options have been proposed. For examples, see Cox et al. (1979), Bunch and Johnson (1992), Broadie and Detemple (1996), Longstaff and Schwarts (2001), Sullivan (2000), and Medvedev and Scaillet (2010). A vast body of empirical finance research has pointed to the existence of jumps in asset time series. Bates (1996) found jumps in the exchange rate process, which can explain the volatility smile in the Deutsche Mark option. Duffie et al. (1999) examined the implication of jumps for option valuation and found that jumps in volatility and price highlight the effect on option "smirks." Johannes (2004) indicated that jumps play an important statistical role in short term interest rate models and pointed out that jumps are important for pricing interest rate options. More empirical studies can be found in Das and Foresi (1996) and Li and Zhang (2014). However, it is difficult to directly extend the abovementioned classical option pricing methods to jump-diffusion situations.

Kou and Wang (2004) and Cai and Kou (2011) provided analytical approximations to price options with double- and mixed-exponential jumps within the Black-Scholes framework. Amin (1993) built a tractable discrete time model for American option valuation by constructing multivariate jumps that are superimposed on the binomial model. Hilliard and Schwartz (2005) developed a robust and flexible bivariate tree approach to option pricing. Ben-Ameur et al. (2016) proposed dynamic programming coupled with finite elements for pricing American-style options in Gaussian and double exponential jumps. However, the application of these approaches beyond the Black-Scholes framework or to different jump distributions for option pricing is somewhat difficult. As pointed out by Hilliard and Hilliard (2015), when extending these approaches beyond the Black-Scholes model, we may need to use the transformation in Nelson and Ramaswamy (1990), which can be complicated if a jump is involved. Beliaeva and Nawalkha (2012) presented mixed jump-diffusion trees under the CEV model. However, the tree method, as we discuss in this paper, is not sufficiently robust when the jump size is not great relative to the diffusion part. Hilliard and Hilliard (2015) chose lattice probabilities by extending the density matching for diffusion to jump diffusions. However, the limitation of this technique is that it requires the diffusion and jump components to be cast as independent state variables.

In this paper, we extend the quadrature method introduced by Andricopoulos et al. (2003) to price American-style options under jump-diffusion models with arbitrary jump distributions, and achieve exceptional accuracy and good speed. The idea is simple and straightforward: we recursively compute the price of Bermudan options, which are close in price to American options, with hundreds of exercisable dates.

Under jump-extended models, the main obstacle in applying the quadrature procedure lies in the transition densities and additional time steps required. Reviewing the work on the quadrature in Sullivan (2000), Andricopoulos et al. (2003), and Chung et al. (2010), we find that when the grid size or time step increases, the computation time grows exponentially because a smaller time step usually needs more quadrature points and the calculation of the node to node takes much more time. To overcome this limitation and reduce the running time, we establish a static point grid and implement the recursive process in a simple matrix manipulation, which leads to a substantial reduction in computation time. We calculate the density numerically, via the convolution integral. Once the densities are calculated, we can recursively value the prices at each time step by comparing the “live value” (non-exercise value) to the “intrinsic value” (exercise value) (Ibanez and Zapatero, 2004).

This paper makes two main contributions to the literature. First, we build the bridge from the diffusion model to the jump case and point out the shortcomings of the mixed-tree method of Beliaeva and Nawalkha (2012) when dealing with small size jumps. Second, we compute the recursive process in matrix form, which means that we manage it by the time step (time to time) based on the static grid instead of the price step (node to node) shown in Andricopoulos et al. (2003) and Sullivan (2000). This also helps to get rid of the Chebyshev approximation and significantly increases the running speed. Specifically, we focus on the popular Gaussian jump model, the double-exponential jump model, and the lognormal jump-extended CEV model. To speed up the computation, we develop more efficient and universal lattice points for the quadrature method than those in Andricopoulos et al. (2003, 2007) and assign probability to these points more reasonably than Beliaeva and Nawalkha (2012), who separate diffusion nodes and jump nodes without overlap. Finally, comparing the efficiency and accuracy of our approach to that of the tree methods and the least squares approach of Longstaff and Schwarts (2001), we find that the quadrature method is robust and the mixed tree method sometimes underestimates values as the maturity increases. Our analysis is easily extended to different jump-diffusion processes and exotic options.

The remainder of this paper is organized as follows. In Section 2, we discuss the basics of the quadrature method, covering topics on the transition density approximation and our suggested static lattice points, and compare it with other methods. Section 3 reports the numerical results of the application of our technique to different jump models. Section 4 concludes.

2. Quadrature methods

2.1. Quadrature method for pricing American options

In this subsection, we briefly introduce the quadrature method for pricing American put options that can be exercised on hundreds of dates. The biggest difference between the quadrature method and the tree method is that the former can correct the "distribution error" (Figlewski and Gao, 1999; Andricopoulos et al., 2003) and allows us to get rid of the transformation in Nelson and Ramaswamy (1990). We follow the framework in Ibanez and Zapatero (2004) and Andricopoulos et al. (2003, 2007) to describe the general algorithm.

Consider a put American option written at time 0 with a spot price X_0 , a strike price K, and maturity date T. The option can be exercised at equally spaced M dates with the time step $\Delta = T/M$. Let $V(X_{n\Delta}, n\Delta)$ be the option price on node $X_{n\Delta}$ at time $n\Delta$.

$$V(X_{n\Delta}, n\Delta) = \max(\exp(-r\Delta)\int_0^{\infty} f(x|X_{n\Delta}) * \mathrm{V}(x, n\Delta + \Delta)dx, \max(0, K - X_{n\Delta})) \quad (1)$$

$n = 0,1,2, \dots, M-1$,

where X is the underlying asset price and $f(x|X_{n\Delta})$ is the transition density from node $X_{n\Delta}$ to x in one time step.

Generally, the abovementioned integral cannot be calculated in a closed form. However, it can be evaluated by quadrature using truncation of domain, where x goes from $\underline{X}_{n\Delta}$ to $\overline{X}_{n\Delta}$ instead of from 0 to positive infinity. $\overline{X}_{n\Delta}$ is the maximum price at which the underlying asset can arrive a later time step, and $\underline{X}_{n\Delta}$ is the minimum price. Sullivan (2000) used Gaussian quadrature to evaluate the risk-neutral expectation for its excellent convergence. Andricopoulos et al. (2003) suggested evaluating it by Simpson's rule, which is based on regularly spaced grids and is more convenient to handle than other options. Considering its universality and robustness, we use Simpson's rule to calculate the integral.

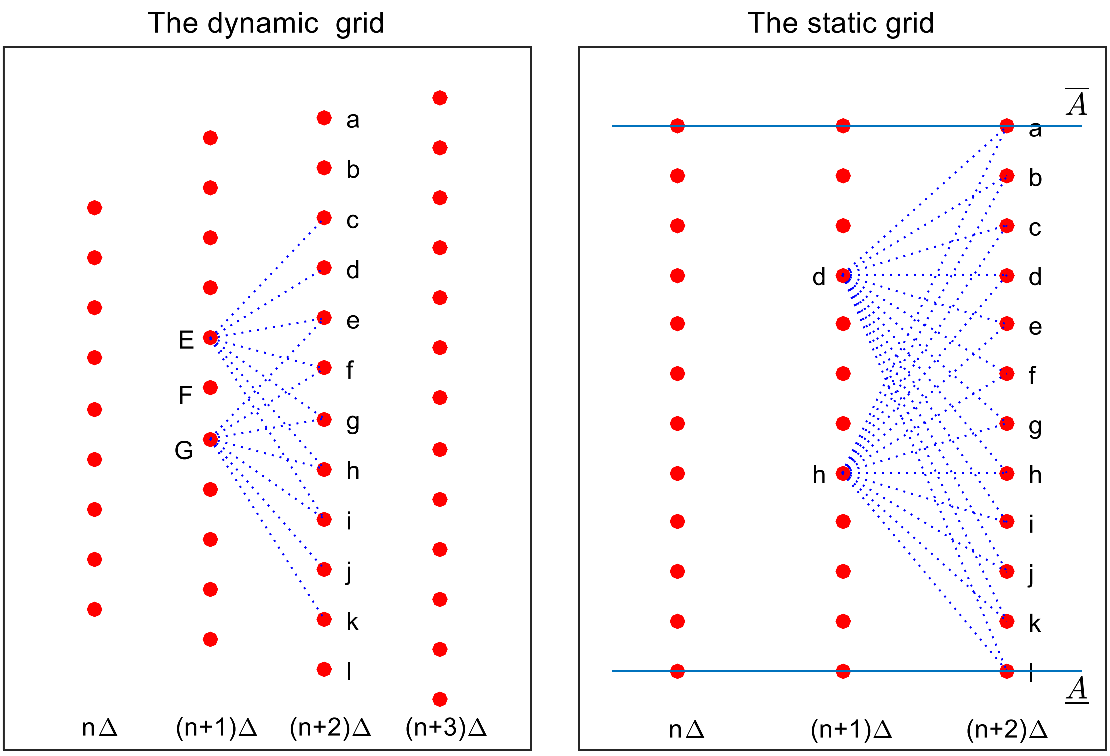
At some exercisable time $n\Delta$, we begin with $G_{n\Delta}$ discretization points $\{X_{n\Delta}^1, X_{n\Delta}^2, \dots, X_{n\Delta}^{G_{n\Delta}}\}$ for $X_{n\Delta}(X_{n\Delta}^i < X_{n\Delta}^{i+1})$. Let $\overline{A}$ be the maximum price at which the underlying asset can arrive before expiration and $\underline{A}$ be the minimum price. In contrast to the "dynamic" (time-varying) lattice points in Andricopoulos et al. (2003), we choose "static" nodes, which means $X_{k\Delta}^i = X_{m\Delta}^i$, $G_{k\Delta} = G_{m\Delta}$, $(0 \leq k < m \leq M)$. See figure 1 for a basic comparison. Let δ be the step size of the price in grids. The vector of lattice points we construct across the exercisable dates is

$$((-I_{min}) : 1 : I_{\max}) * \delta + X_0,$$

where I_{min} is the nearest integer less than or equal to $(X_0 - \underline{A})/\delta$, $I_{\max}$ is the nearest integer greater than or equal to $(\bar{A} - X_0)/\delta$, and $(-I_{min}) : 1 : I_{\max}$ is a regularly spaced vector. Therefore, under this scheme, we have a static grid with the maximum price $\overline{A}_{n\Delta} = \overline{A} = I_{\max} * \delta + X_0$, the minimum price $\underline{A}_{n\Delta} = \underline{A} = X_0 - I_{min} * \delta$, and $X^i = (i - 1 - I_{\min}) * \delta + X_0, 1 \leq i \leq L$ $(L = I_{\max} + I_{\min} + 1)$, here L is the grid size. Wu et al. (2015) adopted a similar static grid in pricing exotic options when some extensive nodes are included.

We build the grid directly on the asset price without the transformation in Nelson and Ramaswamy (1990). Our disproportionately dense grid, on which there are more quadrature points for a higher price, does not influence the results of the different price nodes. Andricopoulos et al. (2003, 2007), Chung et al. (2010), and Chen et al. (2014) used "dynamic" lattice points, which require a repeated computation in each recursion and a longer running time. If we use the suggested "static" points and calculate all of the transition density approximation across the grid points at the beginning, we avoid the need for repeated computation, resulting in a substantial reduction in running time.

Figure 1 Comparison of the dynamic and static grids



Then the following simple equation represents the recursion equation (1) from time $(n+1)\Delta$ to $n\Delta$:

$$V_{n\Delta} = \max\left(\exp(-r\Delta) * \delta * Had(W,P) * V_{(n+1)\Delta}, K - X_{n\Delta}\right), \tag{2}$$

where W represents the weight coefficient matrix with size $L * L$ in applying Simpson's rule, P is an $L * L$ matrix, $P(i,j)$ is the conditional density of $X^j|X^i$ in one time step, and Had refers to the Hadamard product for matrix W and P. $V_{n\Delta}$, whose size is $L * 1$, is the option value of $X_{n\Delta}$ at time $n\Delta$, and $X_{n\Delta}$ presents the grid points $\{X^1, X^2, \dots, X^L\}$.

It is straightforward to calculate the recursion between each interval in a matrix form based on (2) and repeated computation in (2) implements the recursion from time $(n+1)\Delta$ to $n\Delta$ and saves running time compared to node-to-node computation (as in the dynamic grid in Figure 1, we are always valuing the price on node E, then node F, then node G, and so on). This approach also makes it possible to use a smaller time step, and a smaller time step brings the general jump model (which we introduce subsequently) closer to the widely applicable jump model form, especially for a high jump intensity and achieve a tight bound for the optimal exercise frontier.

As indicated in Andricopoulos et al. (2003, 2007), a range of five or six standard deviations away from node $X_{n\Delta}$ during a time step is sufficient in their pure diffusion model, but it may call for a wider range in the jump-extended model when meeting relatively large size jumps. Fortunately, in the matrix form of (2), we can see that it covers the range from $\underline{A}$ to $\bar{A}$ for each node calculated.

2.2. Transition densities under jump-diffusion models

As shown in (1) and (2), transition densities play a vital role in the quadrature method. The calculation of the transition densities represents the gap between the continuous-time models and jump-diffusion models. Here, we start from the general jump-diffusion model,

$$dX_t = \mu(X_t,J)dt + \sigma(X_t,J)dW_t + d\left(\sum_{i=1}^{N_i} h(X_t,J)\right), \tag{3}$$

where N_t is a Poisson process with intensity parameter λ, and $h(X_t,J)$ is the jump function depending on X_t and the jump variable J with the probability density function (PDF) of $\varphi(y|X_t;\theta)$. Obviously, the transition density of the general jump-extended model is different from most of the continuous-time models with discrete observations in which the Euler scheme can be used to approximate the transition density directly, because it is not explicitly computable. Amin (1993) replaced jump distribution with discrete distribution. Hilliard and Hilliard (2015) chose the lattice probability by extending the density matching for diffusions to the density for jump diffusions. However, this setup does not permit state-dependent diffusion volatility for local returns in the jump-diffusion process.

For a tiny time step, we use the following widely applicable form:

$$dX_t = \mu(X_t,J)dt + \sigma(X_t,J)dW_t + h(X_t,J)dN(\lambda), \tag{4}$$

where $N(\lambda)$ is a Bernoulli distribution, which takes the value of 1 with a probability of λdt and the value of 0 with a probability of $1-\lambda dt$. Merton (1976) introduced this Poisson-driven process for the jump process. Beliaeva and Nawalkha (2012) used the approximation by ignoring the diffusion item as follows:

$$dX_t = \begin{cases} \mu(X_t,J)dt + \sigma(X_t,J)dW_t, & \text{with probability } 1-\lambda dt \\ h(X_t,J), & \text{with probability } \lambda dt \end{cases} \tag{5}$$

To be more precise, the PDF in (4) should be

$$dX_t = \begin{cases} \mu(X_t, J)dt + \sigma(X_t, J)dW_t, & \text{with probability } 1-\lambda dt \\ \mu(X_t, J)dt + \sigma(X_t, J)dW_t + h(X_t, J), & \text{with probability } \lambda dt \end{cases}. \tag{6}$$

We can easily calculate the transition density of the first (pure diffusion) part $\mu(X_t, J)dt + \sigma(X_t, J)dW_t$, which we denote as P_0 through the Gaussian approximation scheme, and the second part $\mu(X_t, J)dt + \sigma(X_t, J)dW_t + h(X_t, J)$, which we denote as P_1 through the convolution integral because it is a distribution of the sum of two random variables. Ait-Sahalia (1999, 2002) presented a more explicit sequence of closed-form functions using Hermite polynomials, which handles the diffusion process well, but his work is difficult to extend to the jump-diffusion process. Moreover, the time step we use is less than 0.01 and the Euler scheme produces a similar estimation to that of Ait-Sahalia. If the diffusion part is a normal distribution with PDF $\phi(x)$, and $h(X_t, J)$ has the PDF $p(y, \Delta|X_t;\theta)$, then the second part has the following PDF:

$$P_1(X_{t+1}, \Delta|X_t;\theta) = \int \phi(X_{t+1} - y|X_t;\theta)p(y, \Delta|X_t;\theta)\, dy. \tag{7}$$

By Bayes rule, the transition density is

$$P(X_{t+1}, \Delta|X_t;\theta) = (1-\lambda\Delta)P_0(X_{t+1}, \Delta|X_t;\theta) + \lambda\Delta P_1(X_{t+1}, \Delta|X_t;\theta). \tag{8}$$

For the popular CEV model introduced by Cox and Ross (1976) with the lognormal jump

$$dX_t = rX_t dt + \sigma X_t^{\rho} dW_t + X_t(\exp(J) - 1)dN(\lambda), \tag{9}$$

where X_t is the asset price and $J \sim Normal(\mu_J, \sigma_J^2)$. By the Euler approximation scheme, we have the density

$$P_0(x_{t+\Delta}, \Delta|x_t;\theta) = \frac{1}{\sqrt{2\pi}} \frac{1}{x_t^{\rho}\sigma\sqrt{\Delta}} \exp\left(-\frac{(x_{t+\Delta} - rx_t\Delta - x_t)^2}{2\left(\sigma x_t^{\rho}\right)^2\Delta}\right), \tag{10}$$

and $P_1(x_{t+\Delta}, \Delta|x_t;\theta)$ is expressed in the convolution integral

$$P_1(x_{t+\Delta}, \Delta|x_t;\theta) = \int_{-x_t}^{+\infty} \frac{1}{\sqrt{2\pi\Delta}\sigma x_t^{\rho}} \exp\left(-\frac{(x_{t+\Delta} - x - rx_t\Delta - x_t)^2}{2\left(\sigma x_t^{\rho}\right)^2\Delta}\right) * \frac{1}{\sqrt{2\pi}(x+x_t)\sigma} \exp\left(-\frac{1}{2}\left(\frac{\ln(x/x_t+1) - \mu_J}{\sigma_J}\right)^2\right) dx. \tag{11}$$

Then we get $P(X_{t+1}, \Delta|X_t;\theta)$.

Numerous numerical integration methods are available to calculate the convolution integral in (11), such as the trapezium rule, Simpson's rule, and the Gaussian quadrature. For example, we can get the 0.1% and 99.9% percentiles of the jump variable, establish an equally spaced grid with 100 (200 or more) quadrature points between the percentiles, and then naturally and efficiently implement the integral in Simpson's rule based on the grid.

2.3 Comparisons and extensions

A number of similar methods have been proposed to price American options under jump-extended cases, such as the Gaussian quadrature in Sullivan (2000) and the dynamic programming in Ben-Ameur et al. (2016). Although Sullivan (2000) and Andricopoulos et al. (2003, 2007) both use the quadrature method, they deliver a different efficiency considering the quadrature points used. The Gaussian quadrature needs irregularly spaced grids and requires an exponential increase in effort to calculate the option prices. Despite the function approximation and some extrapolation, the computational effort still increases linearly as the observations M increase and we find that the computation time grows quickly when M increases from 64 to 256 from the numerical results in Sullivan (2000). Moreover, due to the limit of the small M, the reduced form in (4) sometimes does not closely approximate the general jump diffusion model (3). Now we review the recursion in (2) and find that the level of effort stays the same during each time step and is available for a larger M. Furthermore, this helps us get rid of the function approximation and extrapolation, which saves time compared with the framework in Sullivan (2000).

Ben-Ameur et al. (2016) proposed dynamic programming (DP) coupled with finite elements for valuing American-style options under Gaussian and double exponential jumps. The key ingredients for the DP to run are the transition tables. They derive the tables in closed form under the setting of Merton (1976) and Kou (2002). However, their method is difficult to move beyond the GBM (such as CEV) and to different jumps because it seems much more difficult and complex to derive the transition tables. Moreover, the running time grows quickly as the grid size increases (see Table 7 in their work for example).

In addition, estimating the optimal exercise boundary is an important topic in American-style option pricing. Similar to $g(x)$ defined in Andricopoulos et al. (2003), we denote

$$DIF(X_{n\Delta}) = \exp(-r\Delta) * \delta * Had(W, P) * V_{(n+1)\Delta} - \max(K - X_{n\Delta}), \tag{12}$$

which means the difference between the present value of the expected option price value one time step later and the value from an early exercise. The decreasing function is strictly monotonic, so there exists a unique solution $F(n\Delta)$ that meets $DIF\big(F(n\Delta)\big) = 0$. When $V(X_{n\Delta}, n\Delta)$ and $DIF(X_{n\Delta})$ of the grid are calculated recursively, we find a unique number k that holds the inequalities:

$$DIF(X^k) \geq 0, DIF(X^{k+1}) \leq 0.$$

Extrapolation to these estimators can then be performed so that we obtain the solution to the equation $DIF(X_{n\Delta}^*) = 0$, which is the optimal exercise boundary at time $n\Delta$. As a result, we involve the early exercise boundary $F(t)$ on each time step.

Now we consider a monthly (quarterly or weekly) monitored Bermudan option pricing. Slightly different from (2), the recursion goes as follows:

$$V_{n\Delta} = \max(\exp(-r\Delta) * \delta * Had(W,P) * V_{(n+1)\Delta}, K - X_{n\Delta}) \quad \text{when } n = i * M/12 \qquad (13)$$

and

$$V_{n\Delta} = \exp(-r\Delta) * \delta * Had(W,P) * V_{(n+1)\Delta} \quad \text{when } n \neq i * M/12, \qquad (14)$$

where $i = 0,1,\ldots,12$.

Option hedging is another significant topic in option trading, because a delta neutralized portfolio is preferred by investors. A simple and popular approach is to use a finite difference approximation, which requires a reestimation at a spot price of $X_0 + \varepsilon$ and $X_0 - \varepsilon$ (let ε be a number close to zero). Our method seems to be much easier based on the static grid scheme introduced in the previous subsection. We can calculate the prices on nodes d to h in Figure 1 at time 0 (or$n * \Delta$). As mentioned in Sullivan (2002), function approximation can be used to acquire a close and satisfactory fit for these nodes. Thus, hedging coefficients can be easily computed with the fit curve estimated. Here, we select the Chebyshev polynomials as the candidate approximation and the first-order (Delta) and second-order derivative (Gamma) can be derived directly. See Wu et al. (2015) for more details.

3. Numerical study

3.1. The Gaussian jump model

First, we apply the quadrature method to the well-known Black-Scholes model. We begin with this model because numerous studies have been done on it (see Feng and Linetsky (2008), Hilliard and Hilliard (2015), Ben-Ameur et al. (2016) for examples) and there exists an explicit probability density function for this process.

Here, we present the same form of the risk-neutralized version as in Hilliard and Hilliard (2015):

$$\mathrm{d}y = \left(r - \frac{\sigma^2}{2} - \lambda\bar{J}\right)\mathrm{d}t + \sigma dW + \ln(1+J)\, d\ N(\lambda),$$

where $y = \ln(X_{t+1}/X_t)$, $\ln(1+J) \sim N(\gamma', \delta^2)$ and $\bar{J} = E(J), \gamma' = \gamma - \delta^2/2$.

Based on the PDF derived and the lattice points set out in the previous section, the application of the quadrature process to the model is easy and a comparison with other classical approaches can put the quadrature method in perspective.

Table 1 presents the American put option prices from quadrature method and the estimates from the density matching (DM) approach of Hilliard and Hilliard (2015). Type A stands for the equal jump and diffusion volatility coefficient, type B stands for the large relative jump volatility coefficient, and type C stands for the larger relative jump volatility coefficient. Compared with the benchmark from Hilliard and Hilliard (2015), the quadrature method achieves exceptional accuracy, with the errors generally less than 0.01 across the different grid sizes and various strike prices and all of the three types. Moreover, it can be implemented within 1 second for grid sizes less than 2000, and the convergence speed is remarkable. The running time is composed of the transition density computation and recursion process. The former occupies about 60% of the total time. Therefore, the recursion in (2) under the static grid seems efficient.

Table 1 American put option prices under a Gaussian jump diffusion

	A: $\lambda = 5, \gamma = 0, \delta = \sqrt{0.05}, \sigma = \sqrt{0.05}$				**B**: $\lambda = 5, \gamma = 0, \delta = 0.3, \sigma = 0.1$				**C**: $\lambda = 5, \gamma = 0, \delta = \sqrt{0.05}, \sigma = 0.05$			
Strike	DM	QD-5	QD-10	QD-20	DM	QD-18	QD-27	QD-45	DM	QD-16	QD-25	QD-32
30	2.710	2.7156	2.7159	2.7159	4.022	4.0267	4.0267	4.0264	2.262	2.2579	2.2563	2.2563
35	4.592	4.6009	4.6009	4.6009	6.184	6.1866	6.1865	6.1864	3.955	3.9854	3.9848	3.9848
40	7.018	7.0254	7.0254	7.0255	8.771	8.7870	8.7869	8.7868	6.299	6.2958	6.2956	6.2957
45	9.941	9.9471	9.9472	9.9472	11.767	11.7895	11.7895	11.7900	9.154	9.1697	9.1696	9.1696
50	13.307	13.3063	13.3064	13.3065	15.146	15.1450	15.1450	15.1450	12.554	12.5500	12.5498	12.5500
Running time(s)		0.06	0.28	1.10		0.85	1.80	5.00		0.70	1.60	2.50

Note: the spot stock price is 40, the risk free rate is 0.08, and the maturity is 1 year. We set $M = 200$, which maintains the same time step as Hilliard and Schwartz (2015). DM refers to the density matching approach of Hilliard and Hilliard (2015). QD-P means the estimates are from a grid size of P*100, for example, QD-20 means L=20*100=2000. The algorithm is programmed in Matlab and executed on an Intel Core i5-2520 CPU @ 3.4GHz with 8 GB RAM.

3.2. The double-exponential jump model

Now, we consider another popular model, the double exponential jump-diffusion model in Kou (2002), which incorporates the leptokurtic feature. We rewrite the model as follows:

$$\mathrm{d}y = \left(r - q - \frac{\sigma^2}{2} - \lambda\kappa\right)\mathrm{d}t + \sigma dW + \zeta d\ \ N(\lambda),$$

where the PDF of ζ is $f(\mathrm{z}) = p\eta_1 \exp(-\eta_1 z) * \mathbf{1}_{(z\geq 0)} + q\eta_2 \exp(\eta_2 z) * \mathbf{1}_{(z<0)}$. The constants $p, q \geq 0, p + q = 1, \eta_1 > 1, \eta_2 > 0$ and $\kappa = E(\exp(\zeta)) - 1 = p\eta_1/(\eta_1 - 1) + q\eta_2/(\eta_2 + 1) - 1$, $y = \ln(X_{t+1}/X_t)$, r is the risk-free interest rate, and q is the dividend yield. Furthermore, to closely approximate any other random variable, such as the Gamma or Weibull distributions, Cai and Kou (2011) proposed the mixed-exponential jump-diffusion model (MEM), in which the jump size is a weighted average of the exponential distributions. The explicit density over a time interval Δ is easy to derive.

Table 2 provides the monthly monitored Bermudan put option prices from the quadrature method and the estimates from Feng and Linetsky (2008). A basic setting for the models goes as follows:

$$\sigma = 0.1, \lambda = 3, p = 0.3, \eta_1 = 40, \eta_2 = 12, r = 0.05, q = 0.02.$$

We set M equal to 360 and the options can be exercised at $30 * i * \Delta$ $(i = 0,1,2,\ldots,12)$. Again, the quadrature method shows high levels of efficiency and accuracy, such that the errors are generally less than 0.005 across the different grid sizes and strike prices. Furthermore, the quadrature displays consistent convergence to the benchmark and has a satisfactory convergence speed with 1000 or 1500 grids, which seems sufficiently accurate.

Table 2 Bermudan put option prices under a double exponential jump diffusion

Spot	QD-10	QD-15	QD-20	Feng and Linetsky (2008)
85	15.0707	15.0695	15.0695	15.0695
90	11.3660	11.3652	11.3652	11.3662
95	8.5460	8.5453	8.5453	8.5479
100	6.4137	6.4134	6.4134	6.4171
105	4.8182	4.8180	4.8180	4.8225
110	3.6300	3.6298	3.6298	3.6347
115	2.7456	2.7455	2.7455	2.7505
Time(s)	0.42	0.95	1.65	

Note: the strike price is 100 and the maturity is 1 year. QD-P means the estimates are from a grid size of P*100.

Table 3   American put option prices under the lognormal jump-CEV models

	$\lambda = 0$ (pure diffusion)		$\lambda = 2, \mu_J = 0.03, \sigma_J = 0.02$		$\lambda = 4, \mu_J = 0.03, \sigma_J = 0.02$		$\lambda = 2, \mu_J = 0.08, \sigma_J = 0.04$	
Maturity	0.5	1.0	0.5	1.0	0.5	1.0	0.5	1.0
K=80								
Quad-200	1.509	3.456	1.245	2.710	1.030	2.127	1.003	1.986
MixedTree	1.510	3.460	1.228	2.589	1.001	1.941	1.000	1.969
LSM	1.492	3.434	1.234	2.690	1.018	2.111	1.000	1.971
SE	0.012	0.021	0.011	0.018	0.010	0.016	0.010	0.015
K=90								
Quad-200	3.978	6.662	3.419	5.464	2.951	4.506	2.863	4.242
MixedTree	3.984	6.667	3.387	5.269	2.892	4.198	2.861	4.216
LSM	3.975	6.648	3.426	5.468	2.966	4.513	2.878	4.239
SE	0.021	0.030	0.019	0.026	0.017	0.023	0.017	0.022
K=100								
Quad-200	8.211	11.203	7.328	9.577	6.590	8.277	6.412	7.886
MixedTree	8.218	11.211	7.276	9.318	6.497	7.867	6.409	7.855
LSM	8.206	11.178	7.332	9.577	6.575	8.279	6.422	7.874
SE	0.030	0.038	0.027	0.034	0.024	0.031	0.024	0.030
K=110								
Quad-200	14.242	17.039	13.159	15.132	12.311	13.680	12.081	13.231
MixedTree	14.251	17.052	13.100	14.845	12.213	13.249	12.082	13.206
LSM	14.240	17.026	13.165	15.134	12.302	13.671	12.067	13.210
SE	0.036	0.045	0.033	0.040	0.030	0.035	0.029	0.033

This table contains the option prices from the least squares method (LSM) by Longstaff and Schwarts (2001), the mixed tree method by Beliaeva and Nawalkha (2012), and our results from the quadrature method. Quad-200 means $M = 200$. The mixed tree estimates are computed on a 101-node and 500-step tree. SE is the standard error of the LSM estimates.

3.3. The lognormal jump-extended CEV model[1]

The PDF in the two examples above is analytical tractable and these models have attracted sufficient attention in the literature. To further explore the usefulness of the quadrature method, we apply the method to a more general case, the CEV model with lognormal jump, whose PDF is nonanalytic and makes the quadrature method universal and outstanding. Again, based on the PDF in (10) and (11) and the lattice points prepared above, the quadrature method can be easily applied to this model. To test the accuracy and the quadrature method, we compare our approximation with the mixed tree of Beliaeva and Nawalkha (2012) and the least squares methods of Longstaff and Schwarts (2001). Due to a lack of analytic solutions, we treat the LSM estimates as the benchmark. We choose an appropriate parameter setting: $r = 0.03, \sigma = 0.5, \rho = 0.9$ and $X_0 = 100, T = 1$. σ may seem slightly higher because we try to keep the instantaneous variance of this model at the same level as the GBM model, where $\rho = 1(0.3 * 100 = 30, 0.5 * 100^{0.9} \approx 31.5)$.

Table 3 reports the American put option prices for different combinations of jump sizes and intensities. We find that both the jump size and jump intensity have a significant influence on the option valuation and all of the quadrature values are located within 95% confidence intervals of the associated MC values. The bias between the methods for the no jump case $(\lambda = 0)$ and sharp jump case $(\mu_J = 0.08, \sigma_J = 0.04)$ is small within different parameters, moneynesses, and maturities for all three techniques. In the sharp jump case, the jump size is great relative to the diffusion part and the approximation in (5) does not make a big difference to the transition density. Therefore, these methods reach consistent valuations. Now, let us pay attention to the prices generated from the mild jump case $\mu_J = 0.03, \sigma_J = 0.02$. The results from the LSM and quadrature are close to each other, while the mixed tree underestimates the prices. The bias increases as the maturities and jump intensity increase. The estimates from the mixed-tree method are underestimated because the lognormal distribution is skewed to the right and we overestimate the probability on the nodes above the diffusion nodes.

Table 4 shows the computation time and the convergence of the quadrature method under different grid sizes and time steps. We take $\lambda = 4, \mu_J = 0.03, \sigma_J = 0.02$ as an example. We collect the times of the PDF calculation and recursion process separately. We can see that when the time steps increase to 800, it takes about 0.7 seconds to implement the recursion, which shows excellent speed for the quadrature in the static grid. Moreover, it shows consistent convergence to the fourth digit within 2 seconds.

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[1] We consider this model to show the application of the quadrature method to a model whose PDF is nonanalytic and is evaluated from the convolution integral.

Table 4 The convergence in the lognormal-jump extended CEV model

Strike	Quad-2-5	Quad-4-5	Quad-8-5	Quad-2-10	Quad-4-10	Quad-8-10
80	2.1269	2.1266	2.1265	2.1269	2.1267	2.1263
90	4.5056	4.5068	4.5071	4.5057	4.5067	4.5069
100	8.2772	8.2805	8.2816	8.2773	8.2803	8.2813
110	13.6802	13.6859	13.6882	13.6802	13.6857	13.6877
PDF time	0.60	0.49	0.41	2.15	1.55	1.40
Recursion time	0.05	0.09	0.14	0.23	0.38	0.66

Note: the strike price is 100 and the maturity is 1 year. Quad-m-n means the estimates are from time steps of $M = m * 100$ and grid size of $L = n * 100$.

4. Conclusion

In this paper, we extend the quadrature method of Andricopoulos et al. (2003) to American option valuation under jump-diffusion models and attempt to present our approach in a universal manner. We suggest simpler and more robust lattice points for the quadrature method, calculate the transition densities via the convolution integral, and recursively value the option prices in a time step. Moreover, we demonstrate how to price options under the Gaussian jump model, the double-exponential jump model, and the lognormal jump-extended CEV model. Our approach allows for a larger step size and grid size because we implement the recursive process in a matrix manipulation based on the static grid and reach exceptional accuracy. We also find that the bias from the mixed tree method in Beliaeva and Nawalkha (2012) may increase as the jump size or jump intensity vary due to the scheme of probability assigned to the points. Finally, based on the universality of the quadrature method, which can just as easily be applied to exotic options, and the transition density numerically calculated from the convolution integral, we outline more flexible models and jump variables.

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